Real Analysis: Midterm 1 Review

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1 Axioms

This course will almost exclusively use $\mathbb{R}$ and it's subsets. We construct them as follows:

1.1 $\mathbb{N}$

$\mathbb{N}$ is the set of natural numbers, $\{1, 2, 3, \ldots\}$. It has five axioms:

- N1 1 belongs to $\mathbb{N}$
- N2 If n belongs to $\mathbb{N}$, it's "sucessor" n+1 belongs to$\mathbb{N}$.
- N3 1 is not the sucessor of any element in one.
- N4 If n and m have the same sucessor, n=m.
- N5 Any subset S of N which satifies $1 \in S$, and $\forall n \in S, n+1 \in S$, must equal $\mathbb{N}$

These are known as the "Peano" axioms.

1.2 $\mathbb{Z}$

$\mathbb{Z}$ is the natural extension of $\mathbb{N}$ which includes 0 and negative numbers: $\{\ldots$, -3,-2,-1,0,1,2,3,$\ldots\}$

1.3 $\mathbb{Q}$

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\}$.
$\mathbb{Q}$ satifies the "denseness" property, where $\forall a, c \in Q, \exists b$ s.t. $a < b < c$

1.3.1 Rational Zeroes Theorem (RZT)

Let p be an nth degree polynomial with integer coeffiencents $\{c_1, c_2, \ldots c_n\} \subset \mathbb{Z}$ where $c_n, c_0 \neq 0$, and

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0 = 0 \tag{1}$$

Then any rational solutions $x \in \mathbb{Q}$ to this equation must be of the form $\frac{c}{d}$, where $c|c_0$, and $d|c_n$. (Necessary but not sufficient). This theorem allows us to demonstrate a number is NOT rational (paticularly those formed by radicals), by constructing a polynomial for which it is a solution to, then showing that none of the potential rational solutions are an actual solution, or equal to the orginal number. Note that dividing the polynomial by the c_n constant may simplify things significantly.

Any number which is a solution to some form of the polynomial equation above is called "algebraic." (but of course, it is not necarily rational. Note all rational numbers are algebraic, since there is a immediate 1st degree polynomial which they are a solution to).

1.4 $\mathbb{R}$

The set of real numbers and it's properties are what we are largely concerned with. $\mathbb{R}$ is an ordered field, equivalently, a commutative ring with an well-ordering. It satifies the following axioms:

1.4.1 Field Axioms

- A1 $a + (b + c) = (a + b) + c$
- A2 $a + b = b + a$
- A3 $a + 0 = a$
- A4 $\forall a, \exists -a$ s.t. $a + (-a) = 0$
- M1 $a(bc) = (ab)c$
- M2 $ab = ba$
- M3 $1a = a$
- M4 $\forall a \neq 0, \exists a^{-1}$ s.t. $a(a^{-1}) = 1$
- DL $a(b + c) = ab + ac$

1.4.2 Order Axioms

- O1 Either $a \leq b$ or $b \leq a$
- O2 If $a \leq b$ and $b \leq a$, $a = b$
- O3 If $a \leq b, b \leq c$ then $a \leq c$
- O4 If $a \leq b, a + c \leq b + c$
- O5 If $a \leq b, 0 \leq c$, then $ac \leq bc$

1.5 Absolute value and dist(a,b)

We define $|a|$ as:

$$|a| = a \text{ if } a > 0, \text{ and } |a| = -a \text{ if } a < 0. \tag{2}$$

Some properties:

a. $|a| \geq 0$

b. $|ab| = |a||b|$

c. $|a + b| \leq |a| + |b|$

The third item is known as the triangle inequality and is one of the most useful theorems we have. We also have the reverse triangle inequality: $||a| - |b|| \leq |a - b|$

1.6 Boundedness

We say a set of real numbers S is bounded above if $\exists M$ s.t. $\forall s \in S$, $s \leq M$. Similarly, a set of real numbers S is bounded below if $\exists m$ s.t. $\forall s \in S$, $s \geq m$. A set is said to be bounded if it bounded above and below.

1.6.1 Supremum and Infinum

- The supremum of a set is its "Least Upper Bound", that is, an upper bound M where no number less than M is an upper bound.
- The infininum of a set is its "Greatest Lower Bound", that is, a lower bound m where no number greater than m is a lower bound.

We have $\inf S \leq \sup S$. Often we will use these operations on sequences–in this case, we are considering the set of values the sequence takes on.

1.6.2 The Completeness Axiom

Every non-empty subset of the real numbers which is bounded above has a least upper bound. (supremum). Conversely, every subset which is bounded below has a greatest lower bound. (infinum). Note that this is not true of $\mathbb{Q}$.

1.7 $+\infty, -\infty$

We equip the set $\mathbb{R}$ with two symbols $+\infty, -\infty$ which are NOT real numbers, but do satisfy (for all a) $-\infty \leq a \leq +\infty$. Although these are not real numbers and do not have an algebraic structure defined on them, they are still useful, and will enable us to more clearly define certain properties. We say a sequence:

- diverges to $+\infty$ if $\forall M > 0, \exists N$ s.t. $n > N \implies s_n > M$
- diverges to $-\infty$ if $\forall m < 0, \exists N$ s.t. $n > N \implies s_n < m$

and write $\lim s_n = +\infty$ or $\lim s_n = -\infty$.

2 Sequences

2.1 Defition of convergence

A sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers is said to converge if $\exists L \in \mathbb{R}$ such that:

$$\forall\epsilon > 0, \exists N \text{ s.t. } n > N \implies |s_n - L| < \epsilon \tag{3}$$

. In this case, we write $\lim_{n\to\infty} s_n = L$, or $\lim s_n = L$.

2.2 Sequence Limit Theorems

Let $(s_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$ be convergent sequences.

a. Convergent sequences are bounded.

b. $\lim ks_n = k \lim s_n$

c. $\lim s_n + t_n = \lim s_n + \lim t_n$

d. $\lim s_n t_n = (\lim s_n)(\lim t_n)$

e. IF $s_n \neq 0\ \forall n$ and $\lim s_n \neq 0$, $\lim \frac{t_n}{s_n} = \frac{\lim t_n}{\lim s_n}$.

   $\lim \frac{1}{n^p} = 0$ for $p > 0$

   $\lim a^n = 0$ if $|a| < 1$

   $\lim n^{1/n}$=1

   $\lim a^{1/n} = 1$ for $a > 0$

f. Let $(c_n)_{n\in\mathbb{N}}$ diverge to $+\infty$, and $\lim t_n > 0$. Then $\lim c_n t_n = +\infty$.

g. $\lim \frac{1}{s_n} = 0 \iff \lim s_n = +\infty$

2.3 Monotonic Sequences

We define the terms increasing and decreasing as follows:

- A sequence $(s_n)_{n\in N}$ is increasing if $\forall n \in N, s_{n+1} \geq s_n$
- A sequence $(s_n)_{n\in N}$ is decreasing if $\forall n \in N, s_{n+1} \leq s_n$

2.3.1 Theorems

- All bounded monotonic sequences converge.
- Unbounded monotonic sequences diverges to either $+\infty$ or $-\infty$.

Thus $\lim s_n$ is well defined for any monotonic sequence.

In either case such a sequence is said to be monotonic.Note the monotonic property holds $\forall m > n$. A sequence is strictly (increasing, decreasing, or monotone) if the inequality above is instead a strict inequality.

2.4 Cauchy sequences

A cauchy sequence $(s_n)_{n\in\mathbb{N}}$ is one that satifies:

$$\forall \epsilon > 0, \exists N \text{ s.t. } m, n > N \implies |s_n - s_m| < \epsilon \tag{4}$$

Note that the ordering of m and n in the defintion is arbitrary, in fact we may suppose $n > m$ (or vice versa) WLOG if so desired.
A sequence is convergent if and only if it is cauchy. In paticular, all convergent sequences are immediately cauchy, and all cauchy sequences are bounded. An argument shows the lim sup and lim inf of the cauchy sequence are are equal, thus it also converges.

3 Subsequences

A subsequence of a sequence $(s_n)_{n\in\mathbb{N}}$ is another sequence $(s_{n_k})_{k\in\mathbb{N}}$, where $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}$ is a strictly increasing sequence, e.g. $n_1 < n_2 < n_3 < \ldots$. Alternatively the notation $\sigma(n) : \mathbb{N} \to \mathbb{N}$ is used for the "selector" function (it also must be strictly increasing), and the subsequence is denoted $(s_{\sigma(n)})_{n\in\mathbb{N}}$.

For the rest of this section, we let the set S_k be the set of all subsequential limits (the set of limits of all subsequnces) of the considered sequence.

3.1 Properties of subsequences

a. If a sequence (s_n) converges, all it's subsequences converge to that same limit.

b. EVERY sequence has a monotonic subsequence. (Peaks...)

c. Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

d. S_k is non-empty

e. $\lim s_n$ exists $\iff$ S_k has exactly one element (the limit).

3.2 Lim sup and Lim inf

Also here we introduce the closely related ideas of lim sup and lim inf:

$$\limsup s_n := \lim_{N\to\infty} \sup\{s_n : n > N\}$$
$$\liminf s_n := \lim_{N\to\infty} \inf\{s_n : n > N\}$$

These are the supremum or infinum of the set of values of sequence past a certain N. In a sense, we may freely drop the first values of $n \in N$ to more clearly see the bounds of the sequence in the long term.

3.3 Properties of lim sup and lim inf

a. $\limsup s_n = \liminf s_n = \lim s_n \iff s_n$ converges or diverges to $\pm\infty$

b. $\limsup s_n = \sup S_k, \liminf s_n = \inf S_k$