Review Notes: The General Case

Spencer Martz

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1 Systems of 1 st order LDEs

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(t)\boldsymbol{x} = \begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n \\ x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n \\ \dots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n \end{cases} \tag{1}$$

1.1 Solutions

The general solutions of a system of n 1st order LDEs is a sum of n linearly indepedent solution vectors $\boldsymbol{x}_{1\dots n}$.

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n \tag{2}$$

Substitution of intial values will yeild a linear system of n equations for coeffeicents.

1.2 Wronskian of solutions

The wronskian of a set of solution vectors is defined as the determinant of the matrix formed by those solutions as column vectors. By properties of the determinant, the solution vectors are linearly independent iff their determinant is non-zero and linearly dependent if it is zero.

1.3 Constant Coeffecients

For a system with a constant coeffecient matrix $\boldsymbol{P}(t) = \boldsymbol{A}$, solution vectors will be of the form $\boldsymbol{v}_n e^{\lambda t}$, where $\boldsymbol{v}$ and λ are eigenvectors and eigvenvalues repesectively of the coeffeicent matrix. So in order to solve these systems, we calculate eigenvalues and eigenvectors.

If the matrix A has n unique eigenvalues, it will also have n unique eigenvectors. For real eigenvectors, the general solution is simply a linear combonation of $\boldsymbol{v}_{1\dots n}e^{\lambda_{1\dots n}t}$.

For complex eigenvalues, they and their eigenvectors will apear in conjugate form: $\lambda = p \pm qi$, $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}i$. Thus the solution for a paticular eigvenvalue will be

$$x(t) = \boldsymbol{v}e^{(p+qi)t} = (\boldsymbol{a}+\boldsymbol{b}i)e^{pt}(\cos qt + i\sin qt) = e^{pt}(\boldsymbol{a}\cos qt - \boldsymbol{b}\sin qt) + ie^{pt}(\boldsymbol{b}\cos qt + \boldsymbol{a}\sin qt) \tag{3}$$

$$\boldsymbol{x}_1 = e^{pt}(\boldsymbol{a}\cos qt - \boldsymbol{b}\sin qt)$$
$$\boldsymbol{x}_2 = e^{pt}(\boldsymbol{b}\cos qt + \boldsymbol{a}\sin qt)$$

By taking the real and imaginay parts seperately, we get two real valued solutions. The conjugate eigenvector will yeild the same two solutions, so in general:

1. Find a single complex-valued solution associated with the complex eigenvalue .
2. Then find the real and imaginary parts to get two independent real-valued solutions corresponding to the two complex conjugate eigenvalues.

1.3.1 Repeated eigenvalues

For a system with repeated eigenvalues and only one linearly independent eigenvector v_1. The second solution will be of the form

$$\boldsymbol{x}_2 = (\boldsymbol{v}_1 t + \boldsymbol{v}_2)e^{\lambda t} \tag{4}$$

To find v_1 and v_2, first find a (nonzero) solution vector to

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 \boldsymbol{v}_2 = 0 \tag{5}$$

and define v_1 as

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \tag{6}$$

Then one may form the general solutions:

$$c_2 \boldsymbol{v}_1 e^{\lambda t} + c_2(\boldsymbol{v}_1 t + \boldsymbol{v}_2)e^{\lambda t} \tag{7}$$

2 The Laplace Transform

$$F(s) = \mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt \tag{8}$$

The value of the Laplace trasnform is that it may take a differential equation and turn it into a algebraic equation. We may then solve this algebraic equation, do a inverse laplace transform, and obtain a solution. This also has the strength that is does not involve calculating intial values.

2.1 Existance and Uniqueness of the laplace transform

Theorem 1: A function's laplace transform exists (converges) if the function f(t) is peicewise continous, and of exponential order.

1. Piecewise continous: continuous and having a limit at endpoints of a finite number of subintervals which cover the whole interval

2. Exponential Order: A function f(t) is of exponential order if it is bounded above by Me^{ct} for some $M, c, t > T$. All polynomials are of exponential order. More formally:

$$\exists\ M, c, T \in R \text{ s.t. } |f(t)| \leq Me^{ct} \text{ for some t} > T \tag{9}$$

This also means that $\lim s \to \infty\ F(s) = 0$. If two functions f(t) and g(t) satisfy theorem 1, so F(s) and G(s) exist, then if $F(s) = G(s)$ (for some $s > c$, then $f(t) = g(t)$ everywhere f(t) and g(t) are continous.

2.2 Table

function	transform	function	transform
$f(t)$	$F(s)$	$cell2$	$cell3$
$af(t) + bg(t)$	$aF(s) + bG(s)$	$cell5$	$cell6$
$f'(t)$	$sF(s) - f(0)$	$cell8$	$cell9$
$f''(t)$	$s2(F(s)$		

2.3 Unit-Step functions and Dirac-Delta functions