# Review Notes: The General Case

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## 1 Linear 2nd order ODEs

$$Ax'' + Bx' + Cx = D \tag{1}$$

### 1.1 The characteristic equation

For a homogenous 2nd order equation, $Ax'' + Bx' + Cx = 0$, solutions may be obtained by substituting $e^{rx}$ to obtain the characteristic equation $Ar^2 + Br + C = 0$. Since the CEQ is a quadratic equation, it will have two roots $r_1$ and $r_2$. The form of the solution will depend on the roots.

#### 1.1.1 Non-equal real roots $r_1 \neq r_2, \quad r_{1,2} \in R$

The solution takes the form: $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

#### 1.1.2 Equal real roots $r_1 = r_2 = r, \quad r \in R$

The solution takes the form: $(C_1 + C_2 t)e^{rt}$

#### 1.1.3 Complex roots $r_1 \neq r_2, \quad r_{1,2} \in C$

The roots $r_1$ and $r_2$ will be of the form $a \pm bi$, so we may write

$$\begin{gathered}
C_1(e^{at}e^{bit}) + C_2(e^{at}e^{-bit}) \\
= C_1 e^{at}(cos(bt) + isin(bt)) + C_2 e^{at}(cos(bt) - isin(bt)) \\
= (C_1 + C_2)e^{at}cos(bt) + i(C_1 - C_2)e^{at}sin(bt) \\
= C_1 e^{at}cos(bt) + C_2 e^{at}sin(bt) \\
= e^{at}(C_1 cos(bt) + C_2 sin(bt))
\end{gathered}$$

### 1.2 Non-homogenous equations

The solution of a non-homogenous equation will be the sum of it's compliment solution (solution with no homogenous term) and a paticular solution, calculated by the method of undetermined coefficents.

$$x(t) = x_c + x_p \tag{2}$$

## 2 Mass-Spring Systems (Harmonic Oscillators)

Mass spring systems, as well as other harmonic oscillators may be decribed by the 2nd order LDE with const coeff:

$$mx'' + cx' + kx = F(x) \tag{3}$$

Where m = mass, c = damping constant, k = spring contant, and F(x) = damping force. By dividing out m and writing angular frequency $w_0 = \sqrt{\frac{k}{m}}$, and $p = \frac{c}{2m}$, it may be written

$$x'' + 2px' + w_0^2 x = F(x) \tag{4}$$

### 2.1 No driving force – homogenous

#### 2.1.1 Undamped

For an undamped system, the solution will be of the form $Asinw_0t + Bcosw_0t$, which given intial values may be written (for anaylsis purposes):

$$Ccos(w_0 t - \alpha), \text{where } C = \sqrt{A^2 + B^2}, \text{and } \alpha = \begin{cases} \arctan(B/A) : A, B > 0 \\ \arctan(B/A) + \pi : A < 0 \\ \arctan(B/A) + \pi : B < 0, A > 0 \end{cases} \tag{5}$$

| | |
|---|---|
| Amplitude | $C$ |
| Circular Frequency | $w_0$ |
| Phase angle | $\alpha$ |
| Period and Frequency | $\frac{w_0}{2\pi}$ and $\frac{2\pi}{w_0}$ |
| Time lag | $\delta = \frac{\alpha}{w_0}$ |

#### 2.1.2 Damped

The behavior of this equation may be seperated into three cases, determined by the relation of $c^2$ and $4km$. This is becuase this is the discriminant of the CEQ of the general equation. It's value will determine the nature of it's roots – if they are unique, and complex or real

| Underdamped | Critcally Damped | Overdamped |
|---|---|---|
| $c^2 < 4km$ | $c^2 = 4km$ | $c^2 > 4km$ |
| $r_{1,2} \in \mathbb{C} = -p \pm w_1 i$ | $r_1 = r_2 = -p$ | $r_{1,2} \in \mathbb{R}$ |
| $e^{-pt}(A\cos(w_1 t) + B\cos(w_1 t))$ | $e^{-pt}(c_1 + c_2 t)$ | $c_1 e^{r_1 t} + c_2 e^{r_2 t}$ |
| Passes thru eq $\infty$ times | Passes through eq at most once | Passes through eq at most once |

### 2.2 More on underdamped

While the underdamped cases motion is not truly periodic, it is still useful to characterize it as such. Similarly to the undamped cases, we may write the sinusoid component as a single cosine with a phase angle $\alpha$ and amplitude $C = \sqrt{A^2 + B^2}$. The (psuedo) freuqency $w_1$ is given by $\sqrt{w_0^2 - p^2}$.

$$Ce^{-pt}(\cos(w_1 t - \alpha) \tag{6}$$

It has a "time varying amplitude" of $\pm Ce^{-pt}$, and will touch (reach max amplitude) when $w_1 t - \alpha = k\pi$ ($k \in \mathbb{R}$). $w_1$ and $\frac{2\pi}{w_1}$ are called psuedofrequency and psuedoperiod.

## 2.3 Sinusoidal Driving force

### 2.3.1 Undamped

### 2.3.2 Damped