Review Notes: Ordinary Diff EQ Midterm 1

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December 11, 2023

1 Basics of Differential Equation

A differential equation is an equation that relates a function with it's derivatives. They are classified in a number of ways.

Ordinary vs. Partial

An ordinary diff eq. (ODE) is one that only contains the derivatives of a single function.	$4xy'' + e^x y' = x^2$
A partial diff eq. (PDE) is one the contains the derivitaves of multiple function	$y'' + x' - y' = 3e^{-x}$

Linear vs. Non Linear

A linear differential equation is on that is linear in it's depedent variable and derivates	$e^x y'' + cos(x)y' = tan^{-1}x$
A non-linear diff eq. is one the contains the products of derivitaves of it's dependent variable	$y''(y') - \sqrt{y} = 3e^{-x}$

1.1 Solutions

A solution of a diff eq. is a function that (when it and it's derivates are substituted in) solves the equation. A solution may be implicit or explicit. The equation: $y'y = -x, y_0 = r$ has solutions:

$$x^2 + y^2 = r^2 \quad \text{(explicit form)} \tag{1}$$

and,

$$y(x) = \pm\sqrt{r^2 - x^2} \quad \text{(implicit form)} \tag{2}$$

A general solution is a function such that any solution of the differential equation is a linear combonation of the terms of the solution. It is expressed in the form:

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t), \quad c_{1...n} \in R \tag{3}$$

The general solution to an nth order ODE is a linear combonation of n linearly independent solutions.

1.1.1 Equilibrium or constant-valued solutions

An equilibibrium solution is one that is constant valued, $y = k,\ k \in R$. They may be obtained by substitution (so any derivative of y are zero) and solving algrebraicly.

2 Linear First order ODEs and seperable equations

$$y' + P(x)y = Q(x) \tag{4}$$

2.1 Seperable equations

A 1st order ODE is seperable if it may be written:

$$\frac{dy}{dx} = f(x)g(y) \tag{5}$$

And may be solved by dividing by g(y) and integrating both sides

$$\int \frac{1}{g(y)} dy = \int f(x)dx \tag{6}$$

Often this results in a implicit solution, which may be solved in terms of y for an explicit solution.

2.2 The general method of solving 1st order LDEs

1. Step 1: Calculate integrating factor $e^{\int P(x)dx}$
2. Step 2: Multply both sides by integrating factor
3. Step 3: Write left side as derivative of product of functions y(x) and $e^{\int P(x)dx}$
4. Step 4: Integrate both sides
5. Step 5: Divide resultant right-hand side by integrating factor

$$y(x) = e^{-\int P(x)dx} \left[\int Q(x)e^{\int P(x)dx} dx + C \right] \tag{7}$$

3 Models

3.1 Population

The general population equation is given by:

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P \tag{8}$$

where $\beta(t)$ is birth rate, and $\delta(t)$ is death rate. They could be constant or non-constant. Often it is observed that the birthrate decreases (linearly) as population increases. This is written as

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P = aP - bP^2 \tag{9}$$

where a = $\beta_0 - \delta_0$ and b = β_1. If a and b are positive (which they should be), this is known as the logistic equation, and for the purposes of understand it's behavior it best written:

$$\frac{dP}{dt} = kP(M - P) \tag{10}$$

Where k = b and M = a/b. This may be solved in general, with:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}} \tag{11}$$

The behavior of this equation is determined by the initial population and M, or the limiting population. No matter what, as $t \to \infty$, $P(t) = M$.

$P_0 > M$	P tends to M from above
$P_0 = M$	a constant solution
$P_0 < M$	P tends fo M from below

There is another form of this equation, sometimes called the doomsday equation: $P' = kP(P - M)$. In this form, M instead acts as a threshold population, where the population either goes extinct if $P_0 < M$, or explodes (doomsday) for $P_0 > M$.

3.2 Position, Velocity, Acceleration

Position is the integral of velocity, which is the integral of acceleration. For a known force, the position may be caluclated by successive integreation of Newton's 2nd law, $F = Ma$ or $F = mv'$. The constants of integration will be the initial values of velocity and position. A body under constant acceleration (perhaps gravity), without air resistance, a will have a equation of motion:

$$\frac{1}{2}at^2 + v_0 t + x_0 \tag{12}$$

3.2.1 Resistance