Linear Algebra: Midterm 2 Material

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1 Matrix Operations

1.1 Definition of Matrix multiplication

Matrices can be multiplied–that is, if the number of columns in the first bmatrix equals the number of rows in the second. There are two primary ways to think about performating matrix multiplcation, however it is always performed left to right. Note $A^2 = AA, A^3 = AAA$ etc. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \tag{1}$$

1. The "definition" method

$$AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} \tag{2}$$

2. The "row-column" method

$$AB = \begin{bmatrix} * & af + bh \\ ag + ch & * \end{bmatrix} \tag{3}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{mj} \tag{4}$$

1.2 Defintion of Transpose and Inverse of a Matrix

Matrices have two unique operations, transpose and inverse, denoted A^T, and A^{-1} respectively.

1. Transpose of a Matrix: rows become columns, columns become rows.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \tag{5}$$

2. Inverse of a Matrix

   For a (square) bmatrix A, A^{-1} is the unique bmatrix which satifies $AA^{-1} = I$, and $A^{-1}A = I$. The inverse of a bmatrix can be calculated in a number of ways, and not all matrices have an inverse.

1.3 Theorem 1: Properties of Matrix Operations

Basics

1. Associative property of addition

$$A + B = B + C \tag{6}$$

2. Commutative property of addition

$$A + B = B + A \tag{7}$$

Multiplcation

1. Associative property (not commutative)

$$A(BC) = (AB)C = ABC \tag{8}$$

2. Left Distribution over addition

$$A(B + C) = AB + AC \tag{9}$$

3. Right Distribution over addition

$$(B + C)A = BA + CA \tag{10}$$

4. Commutative property of Scalar multiplcation

$$r(AB) = (rA)B = A(rB) \tag{11}$$

Transposition

1. Tranpose identity

$$(A^T)^T = A \tag{12}$$

2. Transpose Distribution over addition

$$(A + B)^T = A^T + B^T \tag{13}$$

3. Tranpose and Scalar Mutlplication associativity

$$r(A^T) = (rA)^T \tag{14}$$

4. Reverse commutative property of tranpose over bmatrix multiplcation

$$(AB)^T = B^T A^T \tag{15}$$

5. Scalar multiplication inversion

$$(rA)^{-1} = r^{-1}A^{-1} \tag{16}$$

Inversion

1. Scalar multiplication inversion

$$(rA)^{-1} = r^{-1}A^{-1} \tag{17}$$

2. Reverse inverse distribution over bmatrix multiplcation

$$(AB)^{-1} = B^{-1}A^{-1} \tag{18}$$

3. Tranverse and inverse communtation

$$(A^{-1})^T = (A^T)^{-1} \tag{19}$$

4. Double inverse property

$$(A^{-1})^{-1} = A \tag{20}$$

2 Matrix Inversion

2.1 The Inverse Matrix Theorem

Let A be an $n*n$ invertible square bmatrix. Then all the following statements are equivalent.

1. A is row-reducible to I
2. A has n pivot positions
3. Ax = 0 has only the trivial solution, x=0
4. The columns of A are linearly independent
5. The transformation $T : R^n \to R^n$ is one-to-one
6. Ax=b has atleast one solution for all B in R^n
7. The columns of A span R^n
8. A^T is invertible

2.2 Finding the inverse of a bmatrix

2.2.1 2x2 matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $ad - bc = det(A) \neq 0$.
Then $A^{-1} = 1/det(A) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

2.2.2 Method one: row reduction to I

Contruct the matrix $A = \begin{bmatrix} A & I \end{bmatrix}$, then row reduce A to I – the augemented I portion of the matrix becomes A^{-1} under the same row operations that transform A to I

2.2.3 Determinant and Adjugate

3 Subspaces

3.1 Defintion of a subspace

A subspace is a set of vectors which contains the zero vector, and is closed under addition and scalar multiplication. Examples include lines and planes thru the origin. Lines and planes not thru the origin are not valid subspaces because they do not contain the zero vector, and the are not closed. Any span of vectors is a valid subspace.

The dimension of a subspace, denoted dim(A), is usually trivial–it is the number of vectors needed to form any basis of a subspace.

3.2 Column and Null Space

These two fundamental spaces are formed by any matrix (when viewed as a linear transformation.)

3.2.1 Column Space

The column space of a matrix is the set of a all linear combinations of the columns of A. When $T : x \to Ax$ is one-to-one, dim(col(A)) will equal the number of columns of A. When $T : x \to Ax$ is not one-to-one, dim(col(A)) will equal the number of linearly independent columns of A.

The rank of a matrix, denoted rank(A) is the dimension of it's column space.

3.2.2 Null space

The null space is the set of all solutions to $Ax = 0$. It is the set of vectors which become the zero vector under the transformation described by A. If $T : x \to Ax$ is one-to-one, then the dimension of the Null space of A is 0. If $T : x \to Ax$ is not one-to-one, dim(nul(A)) is equal to the number of linerly dependent vectors in A

3.3 Rank theorem

For any matrix A with n columns,

$$dim(col(A)) + dim(nul(A)) = n \tag{21}$$

3.4 Invertible Matrices and subspaces

if A is an n by n invertible matrix, then

1. $Col(A) = R^n$
2. $Rank(A) = n$
3. $Nul(A) = \emptyset$
4. $dim(Nul(A) = 0$

3.5 Finding basis for subspaces

Generally amounts to row reduction. To find a basis for a colum space, row reduce to echelon form–the pivot columns are the vectors in the basis for the column space. To find the basis for a null space, row reduce $\begin{bmatrix} A & 0 \end{bmatrix}$ and solve in terms of free variables.

4 Determinants (only square matrices)

4.1 Defintion of a determinant

Determinants be understood and defined in a number of ways. At it's core, it is a function of entries of a matrix that indicates the invertibility of a matrix–if the determinant equals 0, the matrix is not invertible.

Another useful way to understand the determinant is as a measure of the unit region under the transformation defined by A–the "volume distortion" of a given region in R^n under $T : x \to Ax$. When the determinant is zero, the unit region is collapsed by one (or more) dimesions–a cube to sqaure, or a sqaure to a line.

4.2 Calculating the determinant

A_{ij} is defined as the matrix produced by removing the ith row and jth column from A.

$$det(A) = \sum_{j=1}^{n} (-1^{i+j}) a_{ij} det(A_{ij}) \tag{22}$$

$$(-1^{i+j}) \to \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix} \tag{23}$$

The expression $(-1^{i+j}) a_{ij} det(A_{ij})$ is called the (i,j)-cofactor of A, and is denoted C_{ij}.

The determinant of a diagonal matrix (one with non-zero entries only on it's main diagonal) ,or of upper and lower triangular matrices, is the product of their diagonal entries, as each cofactor expansion will result in only one non-zero A_{ij} entry.

4.3 Properties of the derminant

4.3.1 Multiplicative properties of determinants

1. $(detAB) = det(A)det(B) = det(B)det(A) = det(BA)$

4.3.2 Theorem 3: Row operations and determinants

1. If a multiple of one row of A is added to anotherto produce a new matrix B, $det(B) = det(A)$. (Adding or subtracting rows does't affect the determinant)
2. If two rows of A are interchanges to produce B, then $det(B) = -det(A)$. (Row swaps change the sign of the determinant)
3. If one row of A is multipled by k to produce B, then $det(B) = k(det(A))$ (A concenquence of the this the ability to "factor" out a constant from a row)
4. Column operations are also allowed in the same ways, as $det(A) = det(A^T)$

4.3.3 Linear Transformations and Derminants

The area of a region K under the transformation $T : R^n \to R^n, x \to Ax$, is equal to $det(A)$(area of K)

4.3.4 Cramer's rule

For any matrix A, $A_i(b)$ is the matrix produced by replacing the ith column with a vector b.

Let A be invertible. Then the unqiue solution to $Ax = b$ has entries:

$$x_i = \frac{det(A_i(b))}{det(A)} \tag{24}$$

4.3.5 Cramer's rule and inverse matrices

The j'th column of the inverse of an nxn matrix A is the vector vectors x that satifies the equation $Ax = e_i$.

Supose we used cramer's rule to calculate the j'th column of the inverse of a matrix A. To obtain each i'th entry in x, x_i, We would subsiute the j'th vector in I, e_j into the i'th column of A, denoted $A_i(e_j)$.

We could then perform a cofactor expansion down the i'th column, resulting in

$$(-1)^{(i+j)} det(A_{ji}). \tag{25}$$

This is equal to the (j,i)cofactor of A, denoted C_{ji} and so, $A(i,j) = \frac{C(j,i)}{det(A)}$ (note the switched order)

This idea obviously extends to a new method of calculating the inverse matrices

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{1n} & \dots & \dots & C_{nn} \end{bmatrix} \tag{26}$$

or

$$A^{-1} = \frac{1}{det(A)} adj(A) \tag{27}$$

where Adj(A) is the transpose of the cofactor expansion matrix of A.