Integrals: Explained

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1 Multiple Integrals

Multiple integrals can be used to represent a variety of situations, including area and volume, as well as density and other applications. They are often much easier to calculate then they appear. In general, one evalutes them in to out, in the order indicated by the differentials. For example, on a domain $D = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$,

$$\iint_D f(x,y)dA = \int_a^b \int_b^c f(x,y)dydx = \int_a^b (\int_c^d f(x,y)dy)dx \tag{1}$$

1.1 Simple domains

If the domain of integration is rectangular (a cartesian product), the order of integration may be freely interchanged, appropriately changion differentials.This is known as Fubini's Theorem.

1.2 Less simple domains

In the case that the domain of integration is not simply a rectangular region, it may be the case that is it simple on some axis-that is, bounded by a function of the relevant variables on one axis, and by some other describable domain on all others. In that case one may use the functional bounds as the bounds of integration, integrating first with respect to the "simple" axis

In R^2, regions may be either:

- vertically simple: bounded by some $[y = f_1(x)] \leq x \leq [y = f_2(x)]$, with $a \leq x \leq b$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dydx \tag{2}$$

- horzontally simple, bounded by some $[x = g_1(y)] \leq y \leq [x = g_2(y)]$, with $c \leq y \leq d$

$$\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy \tag{3}$$

In R^3, regions may be either x-simple, y-simple, or most commonly z-simple. In that case, for a domain defined by $z_1(x, y) \leq x, y \leq z_2(x, y)$ and $x, y \in D$,

$$\iint_D \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dzdA \tag{4}$$

1.3 Change of Variables–Jacobian

For some regions, paticularly those which are not easily described as given, a process known as "change of variables" may be used to change the variables and perform the integration in a more reasonable domain, while accounting for the change from this process with a matrix known as the Jacobian. For some mapping $G(u,v) = (x(u,v), y(u,v))$ from (uv)-space to (xy)-space, the jacobian is given by:

$$Jac(G) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial (x,y)}{\partial (u,v)} \tag{5}$$

Then the integral over a region $D = G(R)$ may be computed:

$$\iint_D f(x,y)dA = \iint_R f(G(u,v))|Jac(G)|dudv \tag{6}$$

Oftentimes it is difficult to determine the map $G : (u,v) \to (x,y)$, but it is easy to find the inverse mapping, $F : (x,y) \to (u,v)$. In that case,

$$Jac(G) = \frac{1}{Jac(F)} = Jac(F)^{-1} \tag{7}$$

Now, normally you will still need to find G(u,v) to properly map the integrand. But if you are lucky, it may be the case that the integrand may simplify with some factor in the jacobian, and make then become evaluatable without having to actually find G.

1.4 Finding linear maps

1.5 Polar, Cylindrical, Spherical coordinate Integrals

Many times, the domain is more simply described in terms of another coordinate system.

1.5.1 Polar coordiantes

The polar coordinates map, from (r,θ)-space to (xy)-space is

$$G(r,\theta) = (r\cos\theta, r\sin\theta) \tag{8}$$

This takes a point, (r,θ), and maps it to a point (x,y) on the circle with a radius R, and angle θ. As we increase theta, we move around the circle, as we increase r, we move further in or out from the origin. As such, if we have some domain which is circle-ish, like $D = x^2 + y^2 \leq r^2$, it may be useful to change to polar coordinate to evaluate an integral on the simpler (r,θ) domain. For the above example then, and for a general domain D which is "radially simple", we have

$$\iint_D f(x,y)dA = \int_{\theta_0}^{\theta_1} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta \tag{9}$$

1.5.2 Cylindrical Coordiates

Cylindrical cordinates follow quickly from polar, we simply add a z. The differential is the same. For a cylindrically simple domain D:

$$\{(r\cos\theta, r\sin\theta, z) | \theta_0 \leq \theta \leq \theta_1, r_0(\theta) \leq r \leq r_1(\theta), z_0(r,\theta) \leq z \leq z_1(r,\theta)\} \tag{10}$$

$$\iiint_D f(x,y,z)dV = \int_{\theta_0}^{\theta_1} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_0(r,\theta)}^{z_1(r,\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta \tag{11}$$

1.5.3 Spherical Coordinates

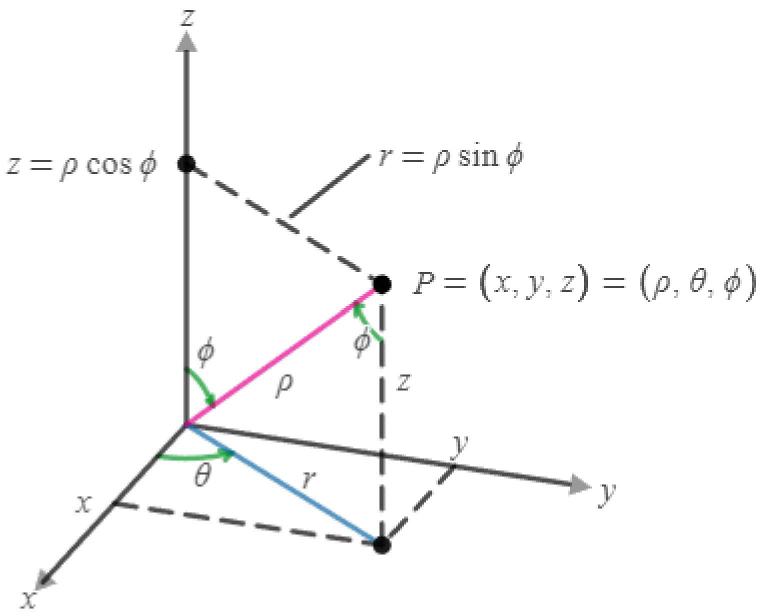
The spherical coordinate map, from (ρ, θ, ϕ) space to (x,y,z) space is:

$$G(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \tag{12}$$

Do not be afraid. It is not as bad as it looks. It is a modification of the cylindrical cordinate system, with $r = \rho \sin\phi$, and $z = \rho\cos\phi$. The differential, dV is given by $dV = \rho^2 \sin\phi \; d\rho d\phi d\theta$ and so for an integral over a spherically simple domain,

$$D = \{G(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) | \theta_0 \leq \theta \leq \theta_1, \phi_0 \leq \phi \leq \phi, \rho_0(\theta, \phi) \leq \rho \leq \rho_1(\theta, \phi)\} \tag{13}$$

$$\iiint_D f(x, y, z) dV = \int_{\theta_0}^{\theta_1} \int_{\phi_1}^{\phi_2} \int_{\rho_0(\theta,\phi)}^{\rho_1(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi) \rho^2 \sin\phi \; d\rho d\phi d\theta$$



2 Line and Surface integrals

2.1 Scalar line integrals

The scalar line integral of f(x,y,z) is defined along a curve C, cutting it into consecutive arcs $C_1, C_2, \ldots, C_N$, picking a sample point $P_i \in C_i$, and forming the riemann sum:

$$\int_C f(x,y,z)ds = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i \tag{14}$$

where Δs_i is the (arc)length of each C_i. Line integrals are evaluated using parametrizations. Suppose $\vec{r}(t)$ directly traverses C from $a \leq t \leq b$ (that is, no stopping or changing direction), and has a continous $\vec{r}\,'(t)$. Then for each $C_i = \vec{r}(t)$ for $t_{i-1} \leq t \leq t_i$, by the arc length forumla we have

$$\Delta s_i = \int_{t_{i-1}}^{t_i} ||\vec{r}\,'(t)||dt \tag{15}$$

Since $\vec{r}'(t)$ is continous, $||\vec{r}'(t)||$ is rougly constant for a small interval, say $\Delta t_i = t_i - t_{i-1}$. Thus the length for each C_i may be approximated as $||\vec{r}\,'(t_i^*)||\Delta t_i$ for some $t_i^* \in \Delta t_i$. So we may take:

$$\lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} f(r(t_i^*))||r'(t_i^*)||\Delta t_i^* \tag{16}$$

(If) the right hand hide convergees to it's respective integral from a to b, then we have:

$$\int_C f(x,y,z)ds = \int_a^b f(\vec{r}(t)||\vec{r}'(t)||dt \tag{17}$$

In essense words, the scalar line integral differential ds becomes $||\vec{r}\,'(t)||dt$. The s is meant to represent arc length;

$$\text{if } s(t) = \int_a^t ||\vec{r}\,'(t)||dt, \text{ then } \frac{\mathrm{d}s}{\mathrm{d}t} = ||\vec{r}\,'(t)|| \text{ (by the FTC).} \tag{18}$$

Thus $ds = \frac{\mathrm{d}s}{\mathrm{d}t}dt = ||\vec{r}\,'(t)||dt$.

2.2 Vector line integrals

Vector line integrals, altough calculated over a more complex field (a vector field), are usually easier then scalar line integrals, since their differential will not involve the $||\vec{r}\,'(t)||$ term. Another important difference is that calculating vector line integral requires a specfied orientation on the curve to be integrated. The VLI is defined as a scalar line intgeral, of the tangential compenent of $\vec{F}$ towards $\vec{T} = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$, the unit tangent vector. Then if $\vec{r}(t)$ directly traverses C from $a \leq t \leq b$, in the correct direction

$$\int_C F \cdot T ds = \int_C F \cdot dr = \int_a^b (F(r(t)) \cdot \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||})||\vec{r}\,'(t)||dt = \int_a^b F(r(t)) \cdot \vec{r}\,'(t)dt \tag{19}$$

The above is a circulation or work integral. There are also flux integrals, which are defined with the dot product of F and unit normal vector. $\vec{N} =< -y'(t), x'(t) >, \vec{n} = \frac{\vec{N}}{||\vec{N}||}$. As before the arc lengths cancel and the forumla is essentially the same:

$$\int_C F \cdot N ds = \int_C F \cdot \vec{n} dr \int_a^b F(r(t)) \cdot \vec{N}\,'(t)dt \tag{20}$$

In either case, if the parametrization is going the opposite direction, the intregral takes on the opposite (negative) value.

2.2.1 Fundamental theorem of conservative vector feilds

For a vector field which is conservative, or path-independent, that is, $\vec{F} = \nabla f$ for some scalar f, then for any path C from P to Q, the value is given:

$$\int_C F \cdot dr = f(Q) - f(P) \tag{21}$$

It follows that for any closed path, the circulation is zero. The proof relies on the chain rule for paths, which states that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\vec{r}(t)) = \nabla f(r(t)) \cdot \vec{r}\,'(t) \tag{22}$$

Thus for some path with parametrizes C from $a \leq t \leq b$, the line integral becomes:

$$\int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)dt = \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} f(\vec{r}(t))dt = f(r(b)) - f(r(a)) \tag{23}$$

2.3 Scalar surface integrals

Let $G(u, v)$ be a function which maps the domain D in xy space to the surface S in R^3. The normal vector $N = G_u \times G_x$ encodes the change in area between D and S. Thus the scalar surface integral is given:

$$\iint_S f \cdot dS = \iint_D f(G(u, v))||N(u, v)||dudv = \iint_D f(G)||N||dA \tag{24}$$

2.4 Vector surface integrals

Similary as above, the vector surface integral is given:

$$\iint_S \vec{F} \cdot dS = \iint_D \vec{F}(G) \cdot \vec{N} dA \tag{25}$$