Review Notes: The General Case

Spencer Martz

April 26, 2024

There are three fundamental theorems we have covered in vector calculus. All are analougous to the fundamental theorem of calculus in that they "cancel out" an integral and a derivative (although the specifics are more complicated).

1 Curves, Surfaces, and Spaces

All therse theorems make use of the orientation of their boundaries, which is important to be explicit about. We start with a simple oriented planar curve–simple, as in non-intersecting, and oriented, as in having a direction of travel specified along the path. We the the region bounded by this path is the region which is on your left if you travel along the curve. The same idea holds for oriented surfaces (those with a continous specified normal vector). For those surfaces with a boundary, we say the boundaries are oriented such that, if you were a normal vector walking on the boundary, the surface would be on your left.

2 Green's Theorem

Green's theorem relates the circulation integral around a simple closed curve in the plane to the integral of the "curl" of the area bounded by the curve. Formally, green's theorem states that for a closed and simply connected domain D, and vector field $F =< F_1, F_2 >$

$$\oint_{\partial D} \vec{F} \cdot dr = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_D curl_z(F) dA \tag{1}$$

3 Stokes' Theorem

Similarly to green theorem, stokes' thorem relates the circulation integral of F around a closed curve ∂S in R^3 to the (flux) integral of the curl over the surface S bounded by it. The formal statement is as follows:

$$\oint_{\partial S} \vec{F} \cdot dr = \iint_S curl(F) \cdot dS \tag{2}$$

This is often useful for planar curves as well as what are known as "solenoidal" or divergence-free vector feilds. These are fields such that

$$\exists \ \vec{A} \mid curl(\vec{A}) = F, \text{or equivalently, } div(F) = 0 \tag{3}$$

The second condition holds since div(curl(A))=0 for any vector field. The vector field A is known as a vector potential, analgous to scalar potentials. Thus if we are given a flux integral over a surface for a solenoidal field, and are able to identify a vector potential A, it may be calculated as the circulation integral around the (approtiately oriented) boundary. This additionally tells us that these solenoidal fields have a property called surface independce, much like line indepences for conservative feilds, which states the flux of F through any surface bounded by a closed curve is equal. Addionally this shows that for any surface which is closed, the flux must be 0, which we will see later.

4 Divergence Theorem

The divergence theorem, also sometimes called Gauss's theorem, state that the flux integral of F on a closed oriented surface is equal to the integral of the divergence of F over the region bounded by the closed surface. The formal statement is as follows:

$$\iint_{\partial W} \vec{F} \cdot dS = \iiint_{W} div(F)dV \tag{4}$$